The Klein-Gordon equation with a generalized Hulthén potential in D-dimensions

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Abstract: An approximate solution of the Klein-Gordon equation for the general Hulthén-type potentials in D-dimensions within the framework of an approximation to the centrifugal term is obtained. The bound state energy eigenvalues and the normalized eigenfunctions are obtained in terms of hypergeometric polynomials.

keyword: Hulthén potential; Klein-Gordon equation; bound states; Approximate analytic solution, Normalization constant, hypergeometric functions, Incomplete Beta function.

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I. INTRODUCTION

The search for exact solutions of wave equations, whether non-relativistic or relativistic, has been an important research area since the birth of quantum mechanics. The generalized Hulthén potential [1]-[18] is given by

$$V(r) = -Z\alpha \frac{e^{-\alpha r}}{1 - qe^{-\alpha r}} \tag{1}$$

where α is the screening parameter and Z is a constant which is identified with the atomic number when the potential is used for atomic phenomena. The Hulthén potential is one of the important short-range potentials which behaves like a Coulomb potential for small values of r and decreases exponentially for large values of r. The Hulthén potential has received extensive study in both relativistic and non-relativistic quantum mechanics [1]-[18]. There is a wealth of literature on the use of the Hulthén potential as an approximation of the interaction potential in a number of area in physics such as nuclear and particle physics [2], atomic physics [3]-[4], solid state physics [5] and chemical physics [6], see also [7] and the references therein. Unfortunately, quantum mechanical equations with the Hulthén potential can be solved analytically only for states with zero-angular momentum [1]-[16]. Recently, some interesting research papers [17]-[18] have appeared to study the l-state solutions of the relativistic Klein-Gordon equation with Hulthén-type potentials. The main idea of their investigation relies on using the approximation of the centrifugal term $\frac{1}{r^2}$ by means of $\frac{1}{r^2} \approx \frac{\alpha^2 e^{-\alpha r}}{(1-e^{-\alpha r})^2}$. Their results show that this approximation is in good agreement with the other methods for small α values. The purpose of the present work is twofold: (i) to extend the l-state approximate solutions [17]-[18] of Klein-Gordon equation with the generalized Hulthén potentials to arbitrary dimension; (ii) to compute the normalization constant of the approximate wave functions that seems to have been overlooked by many researchers [1]-[18].

II. THE KLEIN-GORDON EQUATION IN D DIMENSIONS

The D-dimensional Klein-Gordon equation for a particle of mass M with radially symmetric Lorentz vector and Lorentz scalar potentials, $V(r)$ and $S(r)$, $r = \|\mathbf{r}\|$, is given (in atomic units $\hbar = c = 1$) [19, 20] by

$$\{-\Delta_D + [M + S(r)]^2\}\Psi(\mathbf{r}) = [E - V(r)]^2\Psi(\mathbf{r}), \tag{2}$$

where E denotes the energy and Δ_D is the D-dimensional Laplacian. Transforming to the D dimensional spherical coordinates $(r, \theta_1 \dots \theta_{D-1})$, the variables can be separated using

$$\Psi(\mathbf{r}) = R(r) Y_{l_{D-1},\dots,l_1}(\theta_1 \dots \theta_{D-1}) \tag{3}$$

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where $R(r)$ is a radial function, and $Y_{l_{D-1},\ldots,l_1}(\theta_1 \ldots \theta_{D-1})$ is a normalized hyper-spherical harmonic with eigenvalue $l(l+D-2)$, $l = 0, 1, 2, \ldots$. Thus, we obtain the radial equation of Klein-Gordon equation in D dimensions by substituting Eq.(2) into Eq.(1)

$$\left\{ -\frac{d^2}{dr^2} - \frac{D-1}{r}\frac{d}{dr} + \frac{l(l+D-2)}{r^2} + [M + S(r)]^2 - [E - V(r)]^2 \right\} R(\mathbf{r}) = 0. \tag{4}$$

Writing R as $R(r) = r^{-(D-1)/2}u(r)$ gives

$$\left\{ -\frac{d^2}{dr^2} + \frac{(k-1)(k-3)}{4r^2} + \left[[M + S(r)]^2 - [E - V(r)]^2 \right] \right\} u(r) = 0. \tag{5}$$

where $k = D + 2l$ and $u(r)$ is the reduced radial wave function satisfying $u(0) = 0$.

III. BOUND STATES FOR GENERALIZED HULTHÉN POTENTIAL

We consider the vector and scalar Hulthén potential defined as,

$$V(r) = -\frac{V_0 e^{-\alpha r}}{1 - qe^{-\alpha r}}, \quad S(r) = -\frac{S_0 e^{-\alpha r}}{1 - qe^{-\alpha r}}, \tag{6}$$

where V_0 and S_0 are the depth of the vector and scalar Hulthén potential respectively and α is the screening parameter and $q \neq 0$ is the deformation parameter. Substituting (6) in the Klein Gordon equation (5), we obtain

$$\left\{ -\frac{d^2}{dr^2} + \frac{(k-1)(k-3)}{4r^2} - \frac{(2MS_0 + 2EV_0)e^{-\alpha r}}{1 - qe^{-\alpha r}} + \frac{(S_0^2 - V_0^2)e^{-2\alpha r}}{(1 - qe^{-\alpha r})^2} \right\} u(r) = (E^2 - M^2)u(r). \tag{7}$$

In order to obtain analytic solutions of this equation, we have to use an approximation [17]-[18] for the centrifugal term similar to that used for the non-relativistic cases. We, thus, follow [17]-[18], [21]-[28] and use $\frac{1}{r^2} \approx \frac{\alpha^2 e^{-\alpha r}}{(1-qe^{-\alpha r})^2}$ for the centrifugal term. This approximation is valid for $q = 1$, however, we follow the model used by Qiang *et al* [18]. This allow us to write Eq.(7) as

$$\left\{ -\frac{d^2}{dr^2} - \frac{(2MS_0 + 2EV_0)e^{-\alpha r}}{1 - qe^{-\alpha r}} + \frac{\frac{\alpha^2}{4}(k-1)(k-3)e^{-\alpha r} + (S_0^2 - V_0^2)e^{-2\alpha r}}{(1 - qe^{-\alpha r})^2} \right\} u(r) = (E^2 - M^2)u(r). \tag{8}$$

Eq.(8) can be further simplified using a new variable $z = qe^{-\alpha r}$ $(r \in [0, \infty), z \in [q, 0))$,

$$\left\{ \frac{d^2}{dr^2} + \frac{1}{z}\frac{d}{dz} - \frac{\epsilon^2}{z^2} + \frac{\beta_1}{z(1-z)} - \frac{\frac{1}{4}(k-1)(k-3)}{qz(1-z)^2} - \frac{\beta_2}{(1-z)^2} \right\} u(z) = 0, \tag{9}$$

where we used the dimensionless parameters given by

$$\epsilon = \frac{\sqrt{M^2 - E^2}}{\alpha}, \qquad \beta_1 = \frac{2(MS_0 + EV_0)}{\alpha^2 q}, \qquad \beta_2 = \frac{S_0^2 - V_0^2}{\alpha^2 q^2}. \tag{10}$$

We now look for a solution of (9) in the form

$$u(z) = z^{\epsilon}(1-z)^{\delta} f(z). \tag{11}$$

In this case, Eq.(9) reads

$$\begin{aligned} f''(z) \; &+ \; \left(\frac{1+2\epsilon}{z} - \frac{2\delta}{1-z} \right) f'(z) \\ &+ \; \left(\frac{(\beta_1 - (2\epsilon+1)\delta - \delta^2 + \delta - \beta_2)(1-z) + \delta^2 - \delta - \beta_2 - \frac{1}{4q}(k-1)(k-3)}{z(1-z)^2} \right) f(z) = 0 \end{aligned} \tag{12}$$

We may now choose δ such that

$$\delta^2 - \delta - \beta_2 - \frac{1}{4q}(k-1)(k-3) = 0 \tag{13}$$

which yields

$$\delta = \delta_{\pm} = \frac{1}{2} \pm \frac{1}{2q}\sqrt{q^2(1+4\beta_2) + q(k-1)(k-3)} \tag{14}$$

where $\delta = \delta_+$ for $q > 0$ and $\delta = \delta_-$ for $q < 0$. Thus Eq.(12) reduce to

$$f''(z) = \left(\frac{2\delta}{1-z} - \frac{1+2\epsilon}{z}\right)f'(z) + \left(\frac{4q(\delta(1+2\epsilon) - \beta_1) + (k-1)(k-3)}{4qz(1-z)}\right)f(z) \tag{15}$$

This equation is a special case of a more general differential equation discussed in [29], namely

$$f''(z) = \left(\frac{2az^{N+1}}{1-bz^{N+2}} - \frac{2(m+1)}{z}\right)f'(z) - \frac{wz^N}{(1-bz^{N+2})}f(z), \quad N = -1, 0, 1, \ldots \tag{16}$$

which has exact solutions, for $n = 0, 1, 2, \ldots$, given by

$$f_n(z) = (-1)^n C_n (N+2)^n \left(\frac{2m+N+3}{N+2}\right)_n {}_2F_1(-n, \frac{(2m+1)b+2a}{(N+2)b} + n; \frac{2m+N+3}{N+2}; bz^{N+2}) \tag{17}$$

if

$$w_n(N) = n(N+2)((n(N+2) + (2m+1))b + 2a). \tag{18}$$

Here C_n is the normalization constant and ${}_2F_1(a, b; c; n)$ is a special case [30] of the generalized hypergeometric function

$${}_pF_q(a_1, ..., a_p; c_1, ..., c_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(c_1)_n \cdots (c_q)_n} \frac{z^n}{n!} \tag{19}$$

where the Pochhammer symbol $(a)_n$ is defined by $(a)_n = \Gamma(a+n)/\Gamma(a)$. By putting $N = -1$ and $b = 1$ in (16), using (18), with $a = \delta$ and $w_n(-1) = n(n + 2\epsilon + 2\delta)$, we obtain the energy spectrum of (9) as

$$\begin{aligned} \epsilon_n^{(k)} &= \frac{q[\beta_1 - n^2 - (1+2n)\delta] - \frac{1}{4}(k-1)(k-3)}{2q(n+\delta)} \\ &= \frac{q[\beta_1 - (n+\delta)^2] + q[\delta^2 - \delta - \frac{1}{4q}(k-1)(k-3)]}{2q(n+\delta)} \\ &= \frac{\beta_1 + \beta_2}{2(n+\delta)} - \frac{1}{2}(n+\delta) \end{aligned} \tag{20}$$

where we have used (13). Furthermore, the exact solutions of (12), using (17), now reads

$$f_n(z) = (-1)^n\ C_n\ (1 + 2\epsilon_n^{(k)})_n\ {}_2F_1(-n, 2\epsilon_n^{(k)} + 2\delta + n; 1 + 2\epsilon_n^{(k)}, z) \tag{21}$$

Therefore, we can write the total radial wave function (11) as follows

$$\begin{aligned} u_n(z) &= C_n\ z^{\epsilon_n^{(k)}} (1-z)^\delta\ {}_2F_1(-n, 2\epsilon_n^{(k)} + 2\delta + n; 1 + 2\epsilon_n^{(k)}, z) \\ &= C_n\ z^{\epsilon_n^{(k)}} (1-z)^\delta\ P_n^{(2\epsilon_n^{(k)}, 2\delta-1)}(1-2z) \end{aligned} \tag{22}$$

where we used the definition of Jacobi polynomials [31] given by

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)} {}_2F_1(-n, \alpha+\beta+1+n; 1+\alpha, \frac{1-z}{2}). \tag{23}$$

IV. NORMALIZATION CONSTANT

In this section, we compute the normalization constant C_n appear in (22). As far as we aware, the normalized energy eigenfunctions have not been explicitly worked out in the literature for this case. It is straightforward to note that the normalization constant C_n can be computed using $\int_0^\infty |R^2(r)|r^{(D-1)}dr = \int_0^\infty |u(r)|^2 dr = -\int_q^0 |u(z)^2|\frac{dz}{\alpha z} = 1$ because of the substitution $z = qe^{-\alpha r}$. Thus, we have by mean of (22) that the normalization constant C_n is given by

$$C_n^2 \int\limits_0^q z^{2\epsilon_n^{(k)}-1}(1-z)^{2\delta}\left[{}_2F_1(-n, 2\epsilon_n^{(k)}+2\delta+n; 1+2\epsilon_n^{(k)}, z)\right]^2 dz = \alpha. \tag{24}$$

Using the series representation (19) of the hypergeometric function ${}_2F_1$, being a polynomial of degree n in z, we have

$$C_n^2 \sum_{i=0}^n \sum_{j=0}^n \frac{(-n)_i(2\epsilon_n^{(k)}+2\delta+n)_i}{(1+2\epsilon_n^{(k)})_i i!}\frac{(-n)_j(2\epsilon_n^{(k)}+2\delta+n)_j}{(1+2\epsilon_n^{(k)})_j j!}\int\limits_0^q z^{2\epsilon_n^{(k)}+i+j-1}(1-z)^{2\delta}dz = \alpha. \tag{25}$$

The definite integral in (25) is just the integral representation of Incomplete Beta function [32],

$$B_q(x,y) = \int_0^q t^{x-1}(1-t)^{y-1}dt, \quad \Re(x) > 0, \Re(y) > 0. \tag{26}$$

Therefore, Eq.(25) now reads

$$C_n^2 \sum_{i=0}^n \sum_{j=0}^n \frac{(-n)_i(2\epsilon_n^{(k)}+2\delta+n)_i}{(1+2\epsilon_n^{(k)})_i i!}\frac{(-n)_j(2\epsilon_n^{(k)}+2\delta+n)_j}{(1+2\epsilon_n^{(k)})_j j!} B_q(2\epsilon_n^{(k)}+i+j, 2\delta+1) = \alpha \tag{27}$$

On integrating by parts of (26), we can find that the Incomplete Beta function satisfies the recurrence relation

$$B_q(x+1,y) = \frac{x}{x+y}B_q(x,y) - \frac{q^x(1-q)^y}{x+y}. \tag{28}$$

which in turn can be written in terms of the normalized version of the Incomplete Beta function $I_q(x,y) = B_q(x,y)/B(x,y)$ as

$$\begin{aligned}
I_q(x,y) &= I_q(x-1,y) - \frac{q^{x-1}(1-q)^y}{(x-1)B(x-1,y)} \\
&= I_q(x-2,y) - \frac{q^{x-2}(1-q)^y}{(x-2)B(x-2,y)} - \frac{q^{x-1}(1-q)^y}{(x-1)B(x-1,y)} \\
&= I_q(x-3,y) - \frac{q^{x-3}(1-q)^y}{(x-3)B(x-3,y)} - \frac{q^{x-2}(1-q)^y}{(x-2)B(x-2,y)} - \frac{q^{x-1}(1-q)^y}{(x-1)B(x-1,y)} \\
&= \dots \\
&= I_q(x-m,y) - q^x(1-q)^y\sum_{k=1}^m \frac{q^{-k}}{(x-k)B(x-k,y)}, \quad m = 1,2,\dots
\end{aligned} \tag{29}$$

Thus

$$I_q(2\epsilon_n^{(k)}+i+j, 2\delta+1) = I_q(2\epsilon_n^{(k)}, 2\delta+1) - q^{2\epsilon_n^{(k)}+i+j}(1-q)^{2\delta+1}\sum_{k=1}^{i+j}\frac{q^{-k}}{(2\epsilon_n^{(k)}+i+j-k)B(2\epsilon_n^{(k)}+i+j-k, 2\delta+1)} \tag{30}$$

which allow us to compute the Incomplete Beta function in (27) as $B_q(2\epsilon_n^{(k)}+i+j, 2\delta+1) = B(2\epsilon_n^{(k)}+i+j, 2\delta+1)I_q(2\epsilon_n^{(k)}+i+j, 2\delta+1)$ for $i, j = 0, 1, 2, \dots$. Note, in the case of $i = j = 0$, the sum in (30) is equal to zero.

Although the discussion above for computing the normalization constant assumed that $q \in (0,1)$, the computation for arbitrary $q \neq 0$ can be performed using the analytic expressions [32]:

$$\begin{aligned} B_q(x,y) &= \frac{q^x(1-q)^{y-1}}{x}\sum_{k=0}^{\infty}\frac{(1-y)_k}{(1+x)_k}\left(\frac{q}{q-1}\right)^k \\ &= \frac{q^x(1-q)^{y-1}}{x}\,{}_2F_1(1,1-y;1+x;\frac{q}{q-1}) \end{aligned} \tag{31}$$

for $q \in (-\infty,0)\cup(0,\frac{1}{2})$, and

$$\begin{aligned} B_q(x,y) &= B(x,y) - \frac{q^{x-1}(1-q)^y}{y}\sum_{k=0}^{\infty}\frac{(1-x)_k}{(1+y)_k}\left(\frac{q-1}{q}\right)^k \\ &= B(x,y) - \frac{q^{x-1}(1-q)^y}{y}\,{}_2F_1(1,1-x;1+y;\frac{q-1}{q}) \end{aligned} \tag{32}$$

for $q > \frac{1}{2}$. In the case of $q = 1$, i.e. the Hulthén potential, Eq.(32) yields $B_1(x,y) = B(x,y)$, consequently, Eq.(25) becomes

$$C_n^2\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(-n)_i(2\epsilon_n^{(k)}+2\delta+n)_i}{(1+2\epsilon_n^{(k)})_i i!}\frac{(-n)_j(2\epsilon_n^{(k)}+2\delta+n)_j}{(1+2\epsilon_n^{(k)})_j j!}B(2\epsilon_n^{(k)}+i+j,2\delta+1) = \alpha \tag{33}$$

Using the definition of Beta function [32] in terms of Gamma function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we write (33) as

$$C_n^2\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(-n)_i(2\epsilon_n^{(k)}+2\delta+n)_i}{(1+2\epsilon_n^{(k)})_i i!}\frac{(-n)_j(2\epsilon_n^{(k)}+2\delta+n)_j}{(1+2\epsilon_n^{(k)})_j j!}\frac{\Gamma(2\epsilon_n^{(k)}+i+j)\Gamma(2\delta+1)}{\Gamma(2\epsilon_n^{(k)}+i+j+2\delta+1)} = \alpha \tag{34}$$

By means of the definition of Pochhammer symbols $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we have

$$C_n^2\frac{\Gamma(2\epsilon_n^{(k)})\Gamma(2\delta+1)}{\Gamma(2\epsilon_n^{(k)}+2\delta+1)}\sum_{i=0}^{n}\frac{(-n)_i(2\epsilon_n^{(k)}+2\delta+n)_i(2\epsilon_n^{(k)})_i}{(1+2\epsilon_n^{(k)})_i(2\epsilon_n^{(k)}+2\delta+1)_i i!}\sum_{j=0}^{n}\frac{(-n)_j(2\epsilon_n^{(k)}+2\delta+n)_j(2\epsilon_n^{(k)}+i)_j}{(1+2\epsilon_n^{(k)})_j(2\epsilon_n^{(k)}+i+2\delta+1)_j j!} = \alpha \tag{35}$$

Thus, by using the series representation of the hypergeometric series ${}_3F_2$, again Eq.(19), Eq.(35) then reduce to

$$C_n^2\sum_{i=0}^{n}\frac{(-n)_i(2\epsilon_n^{(k)}+2\delta+n)_i(2\epsilon_n^{(k)})_i}{(2\epsilon_n^{(k)}+2\delta+1)_i i!}\;{}_3F_2\left(\begin{matrix}-n, & 2\epsilon_n^{(k)}+2\delta+n, & 2\epsilon_n^{(k)}+i \\ 1+2\epsilon_n^{(k)}, & 2\epsilon_n^{(k)}+i+2\delta+1 & \end{matrix};1\right) = \frac{\alpha}{B(2\epsilon_n^{(k)},2\delta+1)} \tag{36}$$

which can be used to compute the normalization constant for $n = 0,1,2,\ldots$. In particular, for the ground-state $n = 0$, we have

$$C_0 = \sqrt{\frac{\alpha}{B(2\epsilon_0^{(k)},2\delta+1)}}. \tag{37}$$

V. CONCLUSION

In this work, we have extended the approximate analytic solutions of Klein-Gordon equation with vector and scalar generalized Hulthén potential to arbitrary dimension D. The analytical energy equation and the normalized radial wave functions expressed in terms of hypergeometric polynomials are given. When $D = 3$, our results normalize the approximate analytic solution for bound states obtained in [17] and [18] for Klein-Gordon equation with generalized Hulthén potentials for nonzero angular momentum.

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